Math 101

NOTES ON INVERSES OF FUNCTIONS

Here is a brief discussion on inverses of functions that are covered in Unit 1 of the Math 101 Study Guide.

Definition of the Inverse of a Function

The inverse of a function f is another function, generally designated by f^{-1}, such that the following holds:

1. for all values x in the domain of f:

$$f^{-1} \circ f(x) = I(x) = x \quad \text{OR} \quad f^{-1} \circ f = I$$

and

2. for all values t in the domain of f^{-1}:

$$f \circ f^{-1}(t) = I(t) = t \quad \text{OR} \quad f \circ f^{-1} = I$$

where I is the identity function, defined by $y = I(x) = x$, which maps each real number x to itself.

Definition of the Identity Function

The function I is the identity function with respect to the operation of composition of functions. Therefore, for any function f, the following are true:

1. $f \circ I = f$ OR $(f \circ I)(x) = f(I(x)) = f(x)$ for all x in the domain of f.
2. $I \circ f = f$ OR $(I \circ f)(x) = I(f(x)) = f(x)$ for all x in the domain of f.

This should remind you of the number 1 when multiplying real numbers. In fact, the number 1 is called the multiplicative identity of the real numbers.

EXAMPLE of the properties of the identity function:

Consider the function f defined by $f(x) = 3x^3 - 4x + 1$ for any real number x.

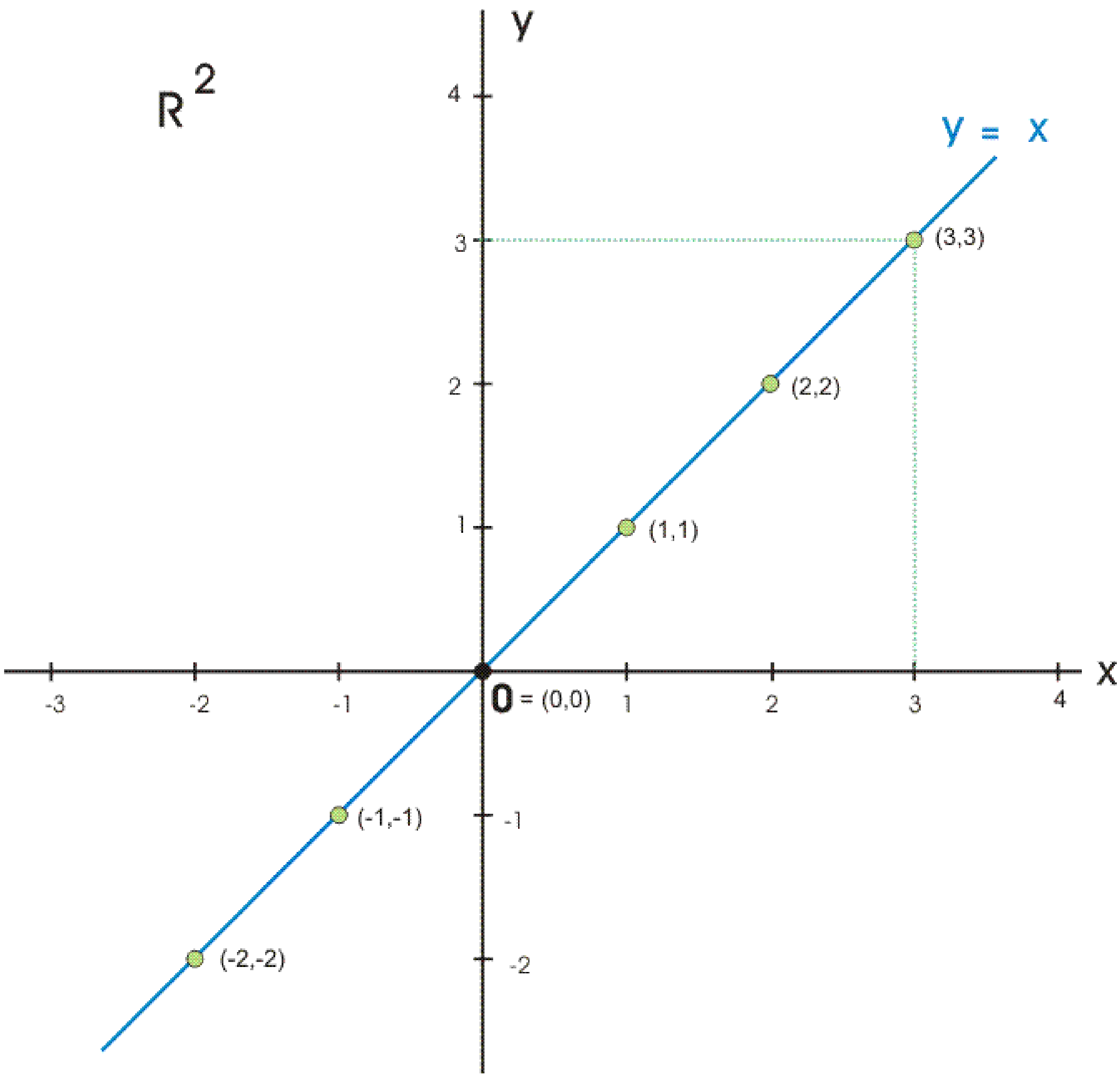
Then $(f \circ I)(x) = f(I(x)) = f(x) = 3x^3 - 4x + 1 = f(x)$
since $I(x) = x$.

and $(I \circ f)(x) = I(f(x)) = I(3x^3 - 4x + 1) = 3x^3 - 4x + 1 = f(x)$
since I maps each real number to itself.

This may seem pedantic, but the role of the identity function is very important in the discussion of an inverse of a function because a function and its inverse must compose together to get the identity function I.

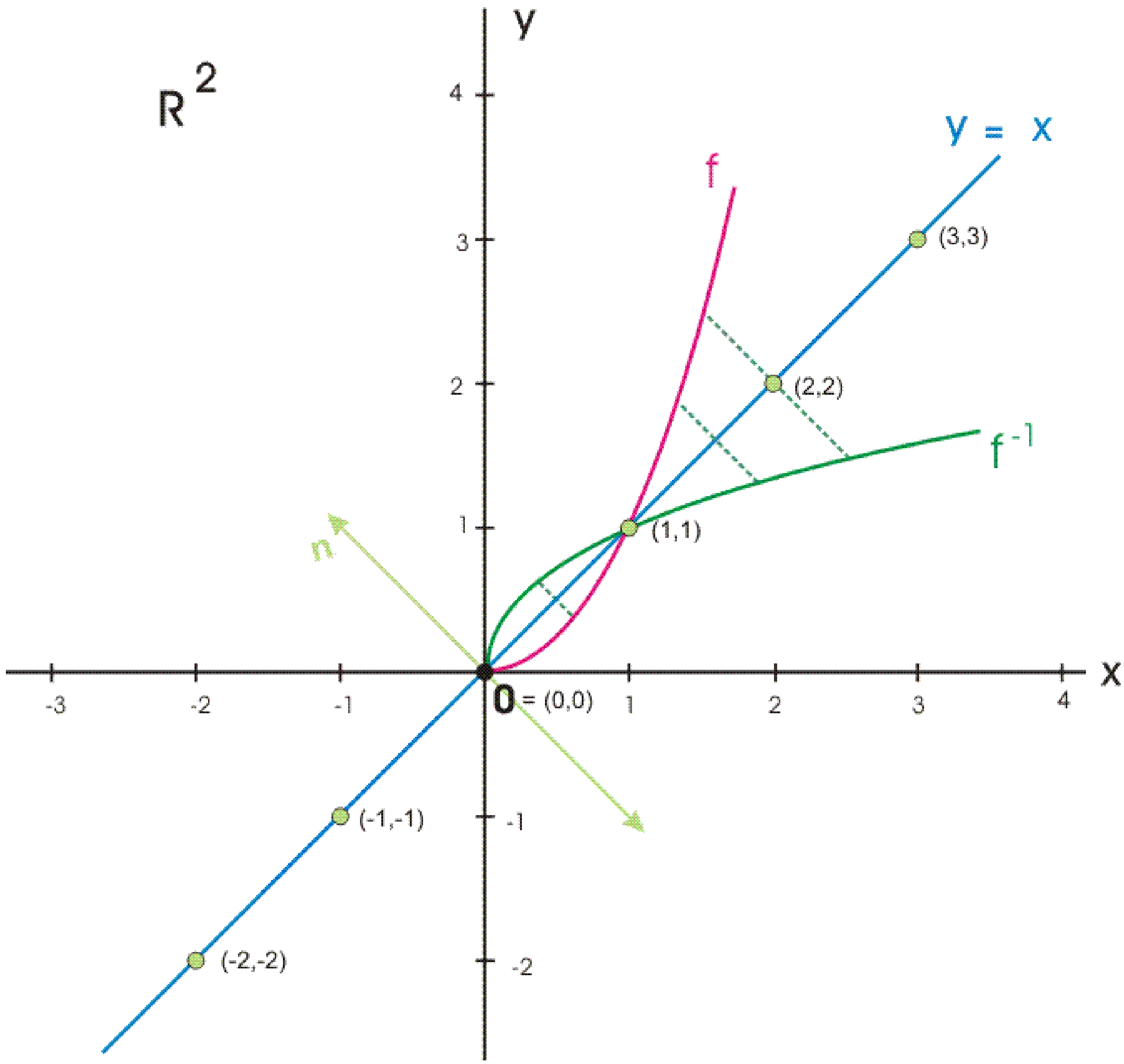
Looking at Inverses of Functions Graphically

NOTE that, because of the way the identity function is defined as $y = I(x) = x$ for all real x, the graph of the identity function I consists of the set of all pairs (x, x) where x is a real number and, when plotted on the xy-plane, consists of the line $y = x$ which is the line passing through the origin at a 45° angle to the positive x-axis.



The commutative algebraic relationships $f^{-1} \circ f = I$ and $f \circ f^{-1} = I$ which must hold for a function f and its inverse f^{-1} imply a kind of symmetry about the identity function *I*. In fact, this is actually the case geometrically. If one plots the graphs of f and f^{-1}, they are orthogonal reflections of one another across the line $y = x$ (the plot of the graph of the identity function *I*) in the *xy*-plane.

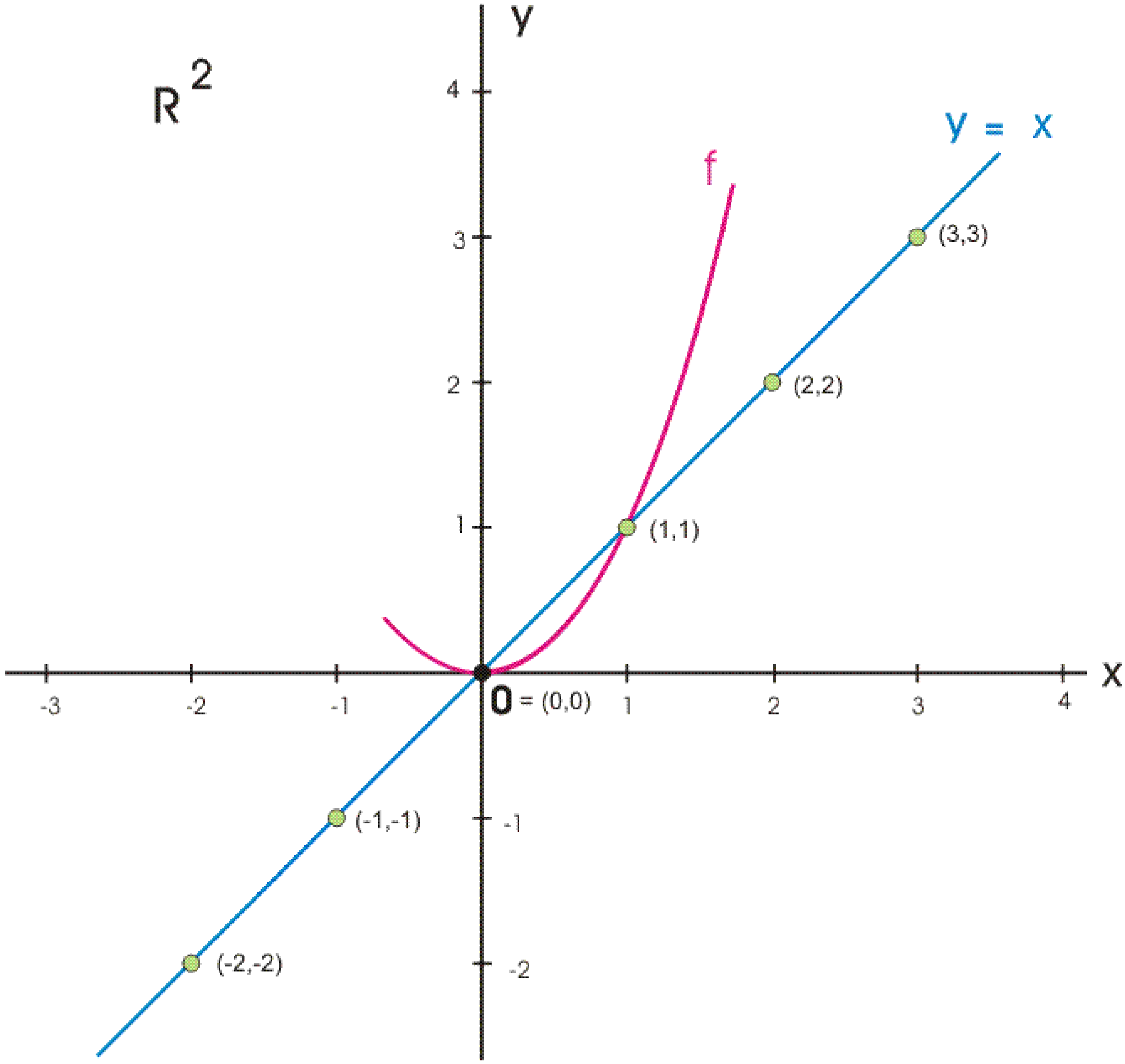
NOTE that, by an *orthogonal* reflection across the line $y = x$, we mean that the reflection is along a line *perpendicular* to the line $y = x$ and each point and its reflected point are *equidistant* to the line $y = x$. In the diagram below, you will see the plots of the graphs of a function f (magenta curve) and its inverse function f^{-1} (green curve).



How can one check if a function has an inverse?

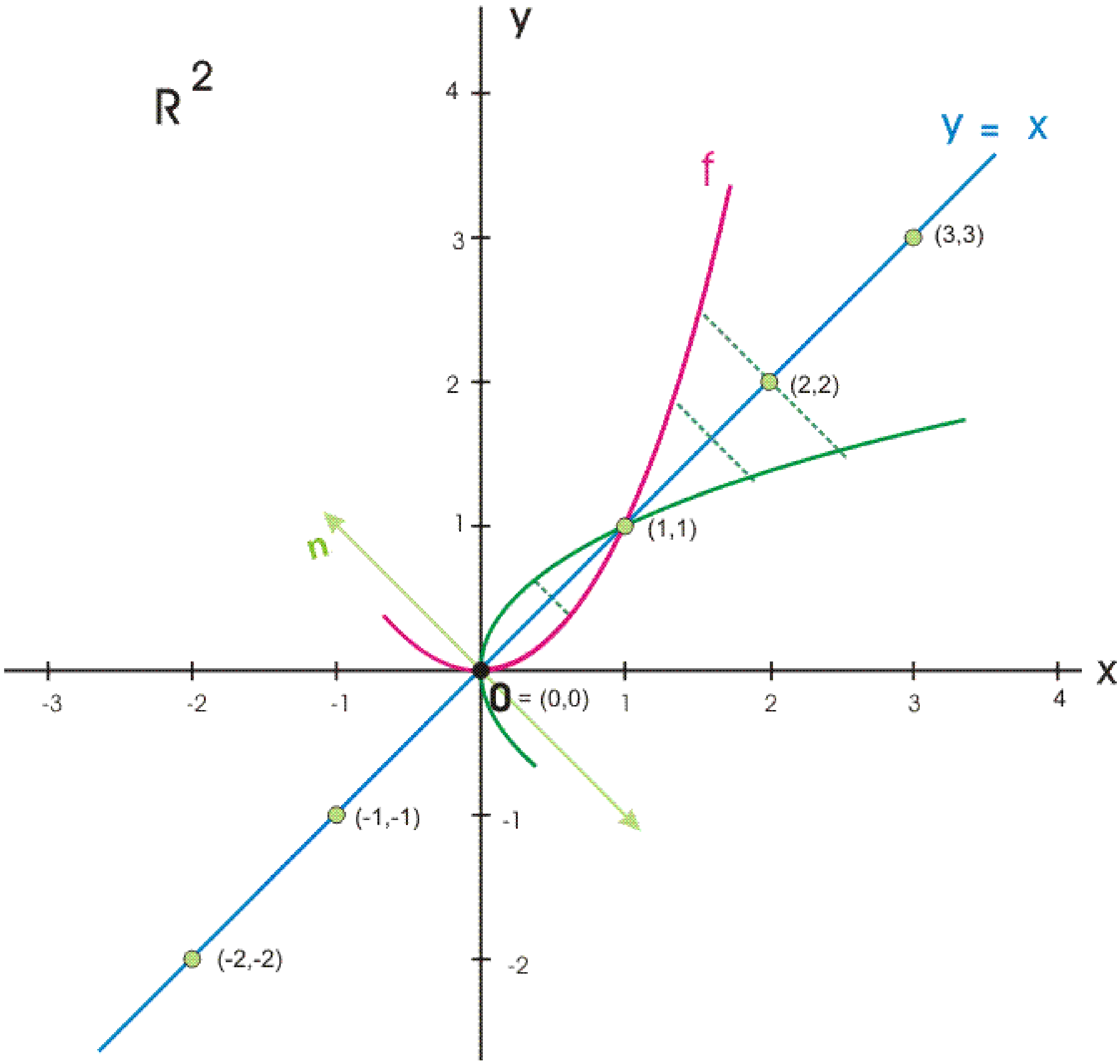
Descriptively, we say that a function f has an inverse if it passes the horizontal line test (HLT) over its domain – in other words, each horizontal line passes through the plot of the graph of f at most once.

Consider the following function f whose graph is plotted below.



This graph does not satisfy the horizontal line test. Horizontal lines y = c for c just above 0 intersect the graph twice.

Consider what would happen if we simply orthogonally reflected the plot across the line $y = x$ to get the graph for its inverse, assuming it existed. We would then get the following.



If we assumed that the inverse of f existed, the green curve would have to be the plot of the graph of f^{-1}. What is the problem then? The green curve cannot qualify as the plot of the graph of a function because it doesn't satisfy the *vertical line test* (VLT)!

This is why the horizontal line test (HLT) criterion is essential for a function f to have an inverse function at all.

How the HLT leads to the Algebraic Criterion for a Function to have an Inverse:

The fact that each horizontal line $y = \text{c}$ intersects the plot at most once implies that
for each y-value c ε Range f, there is **only one** x ε Domain f such that $(x, \text{c}) = (x, f(x))$ ε Graph f (meaning **only one point** on the graph) .

OR

A function satisfies the HLT if
for each y-value c ε Range f, there is **only one** x ε Domain f such that $\text{c} = f(x)$

This leads us to the definition of a function being one-to-one.

Definition of a One-to-One Function:

A function f is said to be one-to-one (or 1-1) if:
for each y ε Range f, there is **only one** x ε Domain f such that $y = f(x)$.

When does a function have an inverse?

Algebraically

A function has an inverse if and only if it is one-to-one.

Geometrically

A function has an inverse if and only if the plot of its graph satisfies the HLT criterion.

Therefore, if TWO distinct values for x are associated by a function f with the same y, the function f is not one-to-one and, therefore, does not have an inverse.

EXAMPLE of determining if a function has an inverse:

Suppose f is the quadratic function defined by $y = f(x) = x^2$ where x is any real number. Then,
For $y = 4$ ε $[0, \infty)$ = Range f, there are TWO domain values $x = 2$ and $x = -2$ such that $y = 4 = f(2) = f(-2)$ – or two domain values 2 and -2 associated by f with 4.

To find them, we simply solve the equation $4 = y = f(x) = x^2$ for x, getting $x = \pm 2$.

Therefore, the function by $y = f(x) = x^2$ is NOT one-to-one and, consequently, does NOT have an inverse.

How does one find the inverse of a function provided it exists?

To find the inverse of a function f, solve the equation $y = f(x)$ for x in terms of y .
You will get an expression looking like: $x = g(y)$.

If x has MORE THAN ONE value for any y-value (does not satisfy the VLT),
the function f is NOT one-to-one and does NOT have an INVERSE function.

If x has ONLY ONE value for each y-value,
the function f is one-to-one and its INVERSE function is $x = g(y)$.

EXAMPLE 1 of finding the inverse of a function:

Let f be the function defined as follows: $y = f(x) = 3x - 2$ for any real number x .
Determine if f has an inverse.

Solve $y = 3x - 2$ for x .
We get: $x = \frac{(y + 2)}{3}$.

Because for each y-value (y = -1, y = 2, y = .5, y = π, etc.) you place into the equation, there is only one corresponding value for x (note that the plot of the graph is a straight line – no down and up curves), the function f is one-to-one and the function

$$x = g(y) = \frac{y + 2}{3}$$ is the inverse function for f.

Sometimes we *switch the variables* x *and* y to get the functional notation we are used to seeing, giving us:

$$y = g(x) = f^{-1}(x) = \frac{x + 2}{3}$$ is the inverse function for f.

We must do this is we want to plot both f and f^{1} on the same xy-plane.
Remember that the horizontal axis is the x-axis (the domain variable axis) and the vertical axis is the y-axis (the range variable axis).

The diagram on page 8 shows both f and f^{1} on the same xy-plane.

EXAMPLE 2 of finding the inverse of a function:

Let f be the function defined as follows: $y = f(x) = x^2$ where $x \geq 0$.
Determine if f has an inverse.

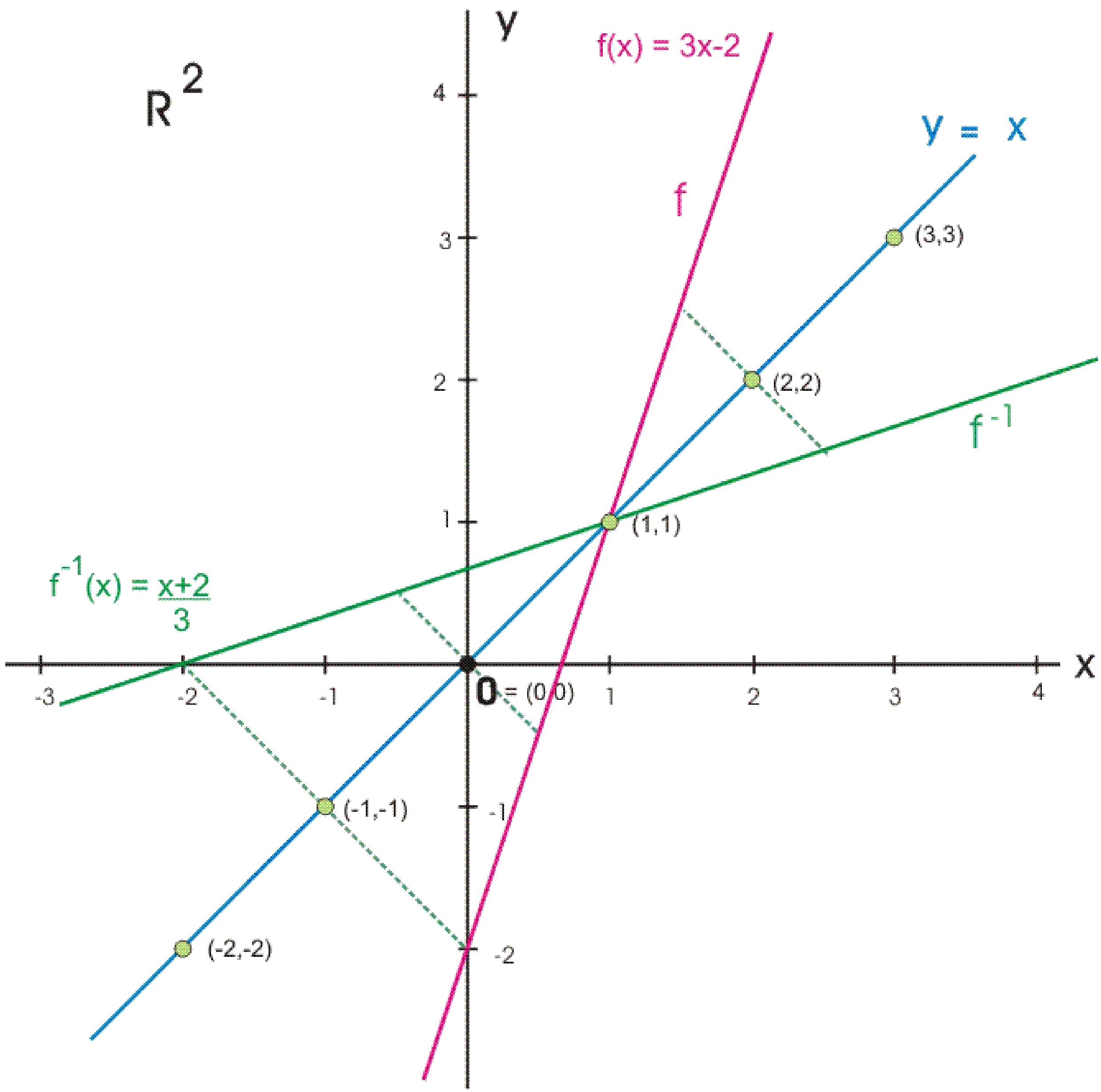
Solving $y = x^2$ for x , we get: $x = \pm\sqrt{y}$. At first, it seems that, for each y-value c where c ≥ 0, there are two values for x, namely $x = \pm\sqrt{c}$, thereby precluding this function from having an inverse. However, if we look more closely at the domain of f , we find that f is defined for only non-negative values of x. Therefore, the solution to the equation becomes: $x = +\sqrt{y}$. This means that the function is, in fact, 1-1 and it has an inverse.

$$x = g(y) = +\sqrt{y}$$ is the inverse function for f.

Switching the variables x and y, we get that $y = g(x) = f^{-1}(x) = \sqrt{x}$ is the inverse function for f.

The diagram on page 9 shows both f and f^{-1} on the same xy-plane.

EXAMPLE 1 of a function *f* and its inverse f^{-1}



EXAMPLE 2 of a function f and its inverse f^{-1}

where $f(x) = x^2$, $x \geq 0$, and $f^{-1}(x) = \sqrt{x}$