Math 101

NOTES ON TRANSFORMATIONS OF FUNCTIONS

Here is a compendium of the various transformations of functions that are covered in Math 101 with some additional explanations on horizontal shifts, stretches and contractions.

RIGID TRANSFORMATIONS

Vertical and Horizontal Shifts:

Suppose $c > 0$. (Note that c must be a positive number.)
Suppose we know what the graph of a function $y = f(x)$ looks like.

To obtain the graph of:
$y = g(x) = f(x) + c$, shift the graph of $y = f(x)$, c units upward (up the y-axis).
$y = g(x) = f(x) - c$, shift the graph of $y = f(x)$, c units downward (down the y-axis).
$y = g(x) = f(x + c)$, shift the graph of $y = f(x)$, c units to the left (left along the x-axis).
$y = g(x) = f(x - c)$, shift the graph of $y = f(x)$, c units to the right (right along the x-axis).

Reflections across an Axis:

To obtain the graph of:
$y = g(x) = - f(x)$, reflect the graph of $y = f(x)$ equidistantly across the x-axis.
$y = g(x) = f(- x)$, reflect the graph of $y = f(x)$ equidistantly across the y-axis.

NON-RIGID TRANSFORMATIONS

Vertical Stretches and Compressions:

Suppose $c > 1$. (Note that c must be a positive number larger than 1.)
Suppose we know what the graph of a function $y = f(x)$ looks like.

To obtain the graph of:
$y = g(x) = c f(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c.
$y = g(x) = (1/c) f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c.

NOTE THAT: VERTICAL compression and stretching occurs with respect to the distance of the point from the x-axis. Points of the original f graph on the x-axis remain fixed in this process.

Horizontal Stretches and Compressions:

Suppose c > 1. (Note that c must be a positive number larger than 1.)
Suppose we know what the graph of a function y = f(x) looks like.

To obtain the graph of:
y = g(x) = f(c x), compress the graph of y = f(x) horizontally by a factor of c.
y = g(x) = f(x / c), stretch the graph of y = f(x) horizontally by a factor of c.

NOTE THAT: HORIZONTAL compression and stretching occurs with respect to the distance of the point from the y-axis. Points of the original f graph on the y-axis remain fixed in this process.

Let us consider four cases in more detail:

CASE 1: Horizontal Compression

To obtain the graph of
y = f(c x) where c >1, compress (shrink) the graph of y = f(x) horizontally by a factor of c, keeping the points (if any) on the y-axis fixed.

WHY?
Consider the following example:

Let f be the function defined as $y = f(x) = x^2$ with domain equal to the interval (-3,1].
Let c = 2 > 1.

Now the graph of $y = x^2$ is a parabola opening upward with vertex at the origin.

Since the domain of f is all numbers x such that $-3 < x \leq 1$, this means that f is only defined for numbers between -3 and 1, excluding -3 but including 1.

Therefore, the graph of f is that part of the parabola defined over the interval $-3 < x \leq 1$.
(See the diagram at the bottom of page 3.)

If c = 2 , then cx = 2x .
Therefore, to find the graph of $g(x) = f(2x) = (2x)^2$, 2x must belong to the domain of f.
In other words, $-3 < 2x \leq 1$ or $-3/2 < x \leq 1/2$ is the domain of g(x) = f(2x).
(Observe that the domain of g is simply the domain of f shrunk by a factor of 2.)

This means that the graph is now fully reproduced over the x-values $-3/2 < x \leq 1/2$.
It has been compressed along the x-axis (horizontally) by a factor of 2.

If you draw it, you will notice that the graph of $f(2x) = (2x)^2 = 4x^2$ is simply a parabola looking like a horizontally compressed version of the graph of f, but the transformed parabola still has vertex at the origin. This shows that the points on the y-axis of the original graph remained fixed in the compression process.

CASE 2: Horizontal Stretching

To obtain the graph of
$y = f(x / c)$ where $c > 1$, stretch the graph of $y = f(x)$ horizontally by a factor of c, keeping the points (if any) on the y-axis fixed.

WHY?

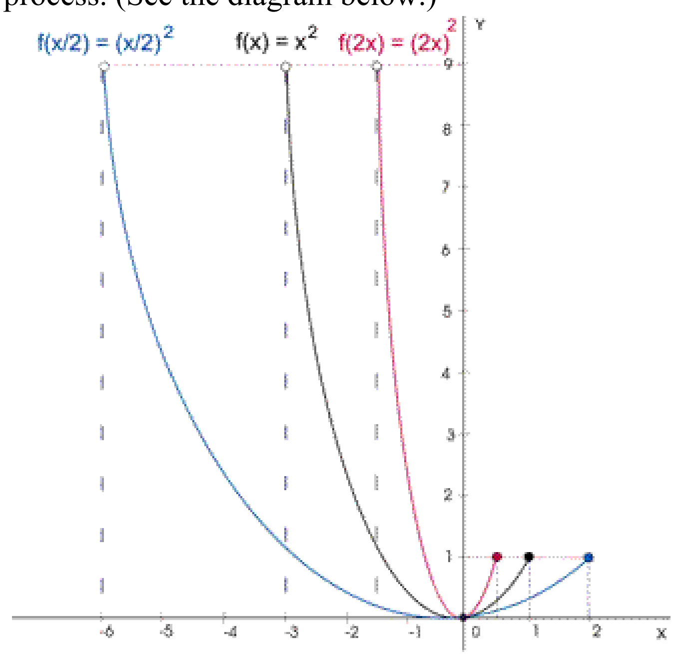
Let f be the function $y = f(x) = x^2$ with domain (-3,1], and let $c = 2 > 1$.

Using an argument similar to the previous example, we know that the graph of $y = x^2$ is a parabola opening upward with vertex at the origin, and the graph of f is that part of the parabola which is defined over the interval $-3 < x \leq 1$.

If $c = 2$, then $x / c = x / 2$.
Therefore, to find the graph of $h(x) = f(x / 2) = (x /2)^2$, $x /2$ must belong to the domain of the function f. This means that $-3 < x / 2 \leq 1$ or $-6 < x \leq 2$ is the domain of $h(x) = f(x / 2)$. (Observe that the domain of h is simply the domain of f stretched by a factor of 2.)

This means that the graph is now fully reproduced over the x-values $-6 < x \leq 2$.
It has been stretched along the x-axis (horizontally) by a factor of 2.

If you draw it, you will notice that the graph of $h(x) = f(x / 2) = (x /2)^2 = x^2 / 4$ is also a parabola looking like a horizontally stretched version of the graph of f, but the transformed parabola still has vertex at the origin. This shows that the points on the y-axis of the original graph remained fixed in the stretching process. (See the diagram below.)



Note that the ranges of all three functions are the same interval [0,9).

CASE 3: Horizontal Translation to the Right

To obtain the graph of
$y = f(x - c)$, where $c > 0$, translate the graph of $y = f(x)$ horizontally to the right by c units.

WHY?
Consider the same function as in the previous two cases where $f(x) = x^2$ with domain (-3,1].
Let $c = 3 > 0$.

If $c = 3$, then $x - c = x - 3$.
Therefore, to find the graph of $k(x) = f(x - 3) = (x - 3)^2$, $x – 3$ must belong to the domain of f.
In other words, $-3 < x - 3 \leq 1$ or $0 < x \leq 4$. (This latter interval is the domain of the function $k(x) = f(x - 3)$ and is the domain of f shifted to the right by 3 units.)

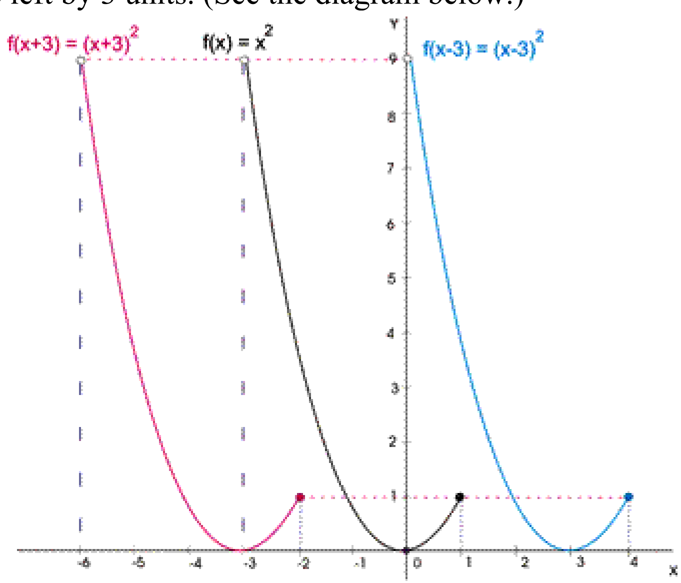
This means that the graph is now fully reproduced over the x-values $0 < x \leq 4$. It has been translated horizontally along the x-axis to the right by 3 units. (See the diagram below.)

CASE 3: Horizontal Translation to the Left

To obtain the graph of
$y = f(x + c)$, where $c > 0$, translate the graph of $y = f(x)$ horizontally to the left by c units.

Similarly, to find the graph of $m(x) = f(x + 3) = (x + 3)^2$, $x + 3$ must belong to the domain of f.
This means that $-3 < x + 3 \leq 1$ or $-6 < x \leq -2$. (This latter interval is the domain of the function $m(x) = f(x + 3)$ and is the domain of f shifted to the left by 3 units.)

Thus the graph of $m(x) = f(x + 3) = (x + 3)^2$ is the same as the graph of f translated horizontally along the x-axis to the left by 3 units. (See the diagram below.)



Note that the ranges of all three functions are the same interval [0,9).