# Math 101

## THE FACTOR THEOREM

### Taking a Closer Look at the Factor Theorem

Recall from page 100 of the Study Guide, Unit 3:

The **Factor Theorem** states the following:

> For a polynomial $P$ of degree $n$ (a positive non-zero integer),
> $P(a) = 0$ where $a$ is any real number **if and only if** $(x - a)$ is a factor of $P(x)$.

**Note that: $(x - a)$ is a factor of $P(x)$ is another way of saying $P(x)$ is divisible by $(x - a)$.

The "if and only if" implication means that it works both ways.

The "if" implication:
**IF** $P(a) = 0$, where $a$ is any real number, **THEN** $(x - a)$ is a factor of $P(x)$.
(the first statement implies the second statement)

AND

The "only if" implication:
**IF** $(x - a)$ is a factor of $P(x)$, where $a$ is any real number, **THEN** $P(a) = 0$.
(the second statement implies the first statement)

### Examples of Using the Factor Theorem

Suppose $P(x) = x^4 - x^2$ . Then $P$ is a polynomial of degree 4. In this case, $n = 4 > 0$.

The "if" implication:

Observe that $P(0) = 0^4 - 0^2 = 0$.
By the Factor Theorem, we can say that $(x - 0)$ or simply $x$ is a factor of $P(x) = x^4 - x^2$ .

Observe that $P(-1) = (-1)^4 - (-1)^2 = 1 - 1 = 0$.
By the Factor Theorem, we can say that $(x - (-1))$ or simply $(x + 1)$ is a factor of $P(x) = x^4 - x^2$ .

The "only if" implication:

If we factor $P(x) = x^4 - x^2$ , we get $P(x) = x^4 - x^2 = x^2 (x^2 - 1) = x^2 (x - 1)(x + 1)$.
By the Factor Theorem, we can deduce that since
$x = (x - 0)$, $(x - 1)$, and $(x + 1) = (x - (-1))$ are all factors of $P(x)$, then
$P(0) = 0$; $P(1) = 0$; and $P(-1) = 0$
– which is evident in this case but not necessarily in all cases.

## Special Cases of the Factor Theorem

See page 103 of the Study Guide, Unit 3.

1. Let $f(x) = x^n - a^n$ where $n$ is a positive non-zero integer and $a$ is any real number.
   Then $(x - a)$ is a factor of $f(x)$.

Simply observe that $f(x) = x^n - a^n$ is a special kind of polynomial of degree $n$.
Since $f(a) = a^n - a^n = 0$, by the "if" implication of the Factor Theorem, we can deduce that $(x - a)$ is a factor of $f(x)$.

2. Let $f(x) = x^n + a^n$ where $n$ is an ODD positive integer and $a$ is any real number.
   Then $(x + a)$ is a factor of $f(x)$.

Again $f(x) = x^n + a^n$ is a special kind of polynomial of degree $n$.
Since $f(-a) = (-a)^n + a^n = -a + a = 0$ (Here we have used the fact that $n$ is an ODD positive integer), by the "if" implication of the Factor Theorem, we can deduce
that $(x - (-a)) = (x + a)$ is a factor of $f(x)$.

## Examples Using the Special Cases of the Factor Theorem

### Example 1:

Determine whether the number $5^6 - 8$ is a prime number by using the Factor Theorem.

Solution:

Recall that a number $p$ is a prime number if and only if its only factors are ± itself or ± 1.

We observe that the number $5^6 - 8 = (5^2)^3 - 2^3$ has the form

$$f(x) = x^n - a^n \text{ where } x = 5^2,\ a = 2, \text{ and } n = 3.$$

Now we **apply the first special case** of the Factor Theorem (listed above) **to the polynomial** $f(x) = x^3 - 2^3$ .

We deduce that $(x - 2)$ is a factor of $x^3 - 2^3$ .

Returning to our number, we recall that $x = 5^2$ .
Therefore $(x - 2) = 5^2 - 2 = 23$ is a factor of $x^3 - 2^3 = (5^2)^3 - 2^3 = 5^6 - 8$.
Since $23 \neq \pm 1$ and $23 \neq \pm(5^6 - 8)$, we conclude that $5^6 - 8$ is not a prime number.

**Example 2:**

Given: $5^p - 1$, where $p$ is an even number (a positive integer divisible by 2).

Prove: that $5^p - 1$ is also an even number.

HINT:

As p is representative of any even number, the proof must involve a general application of the Factor Theorem.

Solution:

Since the number $p$ is divisible by 2, 2 is a factor of $p$, implying that $p = 2 \cdot n$ for some positive integer $n$.

$5^p - 1$ can then be rewritten as: $5^p - 1 = 5^{2 \cdot n} - 1 = (5^2)^n - 1 = (5^2)^n - 1^n$ and has the form $f(x) = x^n - a^n$ where $x = 5^2$ and $a = 1$.

(Here we used the fact that $1 = 1^n$ for any positive integer $n$.)

If we **apply the first special case** of the Factor Theorem **to the polynomial** $f(x) = x^n - a^n$,

we may deduce that $(x - a)$ is a factor of $x^n - a^n$

OR $(5^2 - 1)$ is a factor of $(5^2)^n - 1^n = 5^p - 1$,

showing that 24 is a factor of $5^p - 1$.

If 24 is a factor of $5^p - 1$, then 2 is also a factor of $5^p - 1$.

Thus $5^p - 1$ is an even number. q.e.d. (Latin abbreviation for 'has been proved')

**Example 3:**

Factor $f(x) = x^5 + 32$ into two factors.

Solution:

Observe that $f(x) = x^5 + 32 = x^5 + 2^5$. Here $a = 2$ and $n = 5$ (an ODD positive integer).

Applying **the second special case** of the Factor Theorem **to the polynomial** $f(x) = x^5 + 2^5$, we deduce that $(x + 2)$ is a factor of $f(x)$.

We now divide $(x + 2)$ into $(x^5 + 32)$ using polynomial division.

$$
\begin{array}{r|l}
 & x^4 - 2\cdot x^3 + 4\cdot x^2 - 8\cdot x + 16 \\
\hline
x + 2 & x^5 + 0\cdot x^4 + 0\cdot x^3 + 0\cdot x^2 + 0\cdot x + 32 \\
 & \underline{x^5 + 2\cdot x^4} \\
 & -2\cdot x^4 + 0\cdot x^3 \\
 & \underline{-2\cdot x^4 - 4\cdot x^3} \\
 & +4\cdot x^3 + 0\cdot x^2 \\
 & \underline{+4\cdot x^3 + 8\cdot x^2} \\
 & -8\cdot x^2 + 0\cdot x \\
 & \underline{-8\cdot x^2 - 16\cdot x} \\
 & +16\cdot x + 32 \\
 & \underline{+16\cdot x + 32} \\
 & 0
\end{array}
$$

Therefore, $f(x) = x^5 + 32 = (x + 2) \cdot (x^4 - 2\cdot x^3 + 4\cdot x^2 - 8\cdot x + 16)$.