Unit 5
Double and Triple Integrals

In this unit, we develop several methods for evaluating multiple integrals, including reduction to repeated integrals, transformation to cylindrical coordinates, and transformation to spherical coordinates. We also apply multiple integrals to finding areas and volumes.

Note: Unit 5 is based on Chapter 16 of the textbook*, Salas and Hille's Calculus: Several Variables*, 7th ed., revised by Garret J. Etgen (New York: Wiley, 1995). All assigned readings and exercises are from that textbook, unless otherwise indicated.

Objectives

Detailed objectives are given in each of the sections listed below.

1. Multiple-sigma Notation
2. The Double Integral over a Rectangle
3. The Double Integral over a Region
4. Evaluating Double Integrals by Repeated Integrals
5. The Double Integral as the Limit of Riemann Sums; Polar Coordinates
6. Applications of Double Integrals
7. Triple Integrals
8. Reduction to Repeated Integrals
9. Cylindrical Coordinates
10. The Triple Integral as the Limit of Riemann Sums; Spherical Coordinates
11. Jacobians and Changing Variables in Multiple Integrations

Objective 1 After completing this section, you should be able to

a. evaluate sums involving double and triple sigma notation.

b. demonstrate certain conclusions relating to multiple sigma notation.

Reading

Read Section 16.1, pages 1033-1036.

Exercises

Complete problems 1-10 and 17-20 on page 1036.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

sigma notation $\sum_{i=1}^{m} a_i$, $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$ and $\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{q} a_{ijk}$

factoring a constant through a double (or triple) sum; for example,
$\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha a_{ij} = \alpha \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$

Skills

Before you proceed to Objective 2, make certain that you can meet each of the sub-objectives listed under Objective 1.

Objective 2 After completing this section, you should be able to

a. find the upper sum and the lower sum of P for function on a rectangle, given values for the partitions P_1 and P_2.

b. evaluate simple double integrals, given the upper sum and the lower sum.

c. demonstrate certain conclusions relating to double integrals.

Reading

Read Section 16.2, pages 1036-1043.

Exercises

Complete all odd-numbered problems on pages 1043-1044.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

double integral over a rectangle $\iint_R f(x,y)dx\,dy$

partition

upper sum

lower sum

relation between double integral and volume

Skills

Before you proceed to Objective 3, make certain that you can meet each of the sub-objectives listed under Objective 2.

Objective 3 After completing this section, you should be able to

a. calculate the volume of a solid using double integrals.

b. demonstrate certain conclusions relating to double integrals.

Reading

Read Section 16.3, pages 1044-1048.

Exercises

Complete all problems on page 1048.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

basic region Ω

double integral over a region Ω $\iint_\Omega f(x,y)dx\,dy = \iint_R f(x,y)dx\,dy$

double integral as the volume of a solid

elementary properties of the double integral (linearity, order, additivity, mean-value condition)

mean-value theorem for double integrals

g-weighted average of f on Ω

Skills

Before you proceed to Objective 4, make certain that you can meet each of the sub-objectives listed under Objective 3.

Objective 4 After completing this section, you should be able to

a. determine whether a given region is a type I region or a type II region.

b. evaluate double integrals on regions of type I or type II.

c. solve practical problems using double integration.

d. demonstrate certain conclusions relating to double integration.

Reading

Read Section 16.4, pages 1048-1059.

Exercises

Complete all of the odd-numbered problems 1-53 on pages 1059-1060.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

type I region

reduction formula for type I region $\iint_{\Omega} f(x,y)dx\ dy = \int_a^b \left(\int_{\Phi_1 x}^{\Phi_2 x} f(x,y)dy \right) dx$

type II region

reduction formula for type I region $\iint_{\Omega} f(x,y)dx\ dy = \int_c^d \left(\int_{\Psi_1 y}^{\Psi_2 y} f(x,y)dx \right) dy$

repeated integration

Skills

Before you proceed to Objective 5, make certain that you can meet each of the sub-objectives listed under Objective 4.

Objective 5

After completing this section, you should be able to

a. solve area and volume problems expressed in polar coordinates using double integrals.

b. change problems involving rectangular coordinates to problems in polar coordinates, and solve them using double integrals.

Reading

Read Section 16.5, pages 1061-1067.

Exercises

Complete odd-numbered problems 1-17 and problems 18-35 on pages 1067-1068.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

diameter of a set

double integral as the limit of a Riemann sum

area of a region expressed as the double integral of polar coordinates

Skills

Before you proceed to Objective 6, make certain that you can meet each of the sub-objectives listed under Objective 5.

Objective 6

After completing this section, you should be able to use double integration to solve problems involving the mass of a plate, the centre of mass of a plate, the centroid of a region, the moment of inertia of a plate, and the radius of gyration of an object.

Reading

Read Section 16.6, pages 1068-1074.

Exercises

Complete all odd-numbered problems on pages 1074-1075.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

plate

mass of a plate

centre of mass of a plate

centroid

axis of rotation

kinetic energy formula

moment of inertia

rotational inertia

moment of inertia of a plate

radius of gyration

parallel axis theorem

Skills

Before you proceed to Objective 7, make certain that you can meet Objective 6.

Objective 7 After completing this section, you should be able to

a. find the upper sum and the lower sum of P for function on a rectangular solid, given values for the partitions P_1, P_2 and P_3.

b. solve mass, centre of mass, centroid, and moment of inertia problems involving three-dimensional objects, using triple integration.

c. demonstrate various conclusions relating to triple integrals.

Reading

Read Section 16.7, pages 1075-1081.

Exercises

Complete problems 1-11 on pages 1081-1082.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

triple integral

triple integral over a rectangular solid

partition of an angular solid

upper sum

lower sum

triple integral over a basic solid

relationship between volume and the triple integral

properties of the triple integral (linearity, order, additivity, mean-value condition)

average value

weighted average

g-weighted average of a function on a basic solid

mass formula for a three-dimensional object

centre of mass formula for a three-dimensional object

moment of inertia formula for a three-dimensional object

homogeneous three-dimensional object

Skills

Before you proceed to Objective 8, make certain that you can meet each of the sub-objectives listed under Objective 7.

Objective 8

After completing this section, you should be able to

a. evaluate repeated integrals.

b. use triple integration to find the volume of a solid, its mass, its centre of mass and its moment of inertia.

c. express properties of three-dimensional solids as repeated integrals.

d. and demonstrate certain conclusions relating to repeated integrals.

Reading

Read Section 16.8, pages 1082-1089.

Exercises

Complete odd-numbered problems 1-21, and problems 22-35, 39-41, 50, 51, 53 and 54 on pages 1089-1091.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

triple integral over a basic solid expressed by three ordinary integrals

$$\iiint_T f(x,y,z)dx\ dy\ dz = \int_{a_1}^{a_2}\int_{\Phi_1(x)}^{\Phi_2(x)}\int_{\Psi_1(x,y)}^{\Psi_2(x,y)} f(x,y,z)dz\ dy\ dx$$

alternative orders of integration

Skills

Before you proceed to Objective 9, make certain that you can meet each of the sub-objectives listed under Objective 8.

Objective 9

After completing this section, you should be able to

a. convert an equation expressed in rectangular coordinates into one expressed in cylindrical coordinates.

b. convert an equation expressed in cylindrical coordinates into one expressed in rectangular coordinates.

c. evaluate triple integrals of cylindrical coordinates.

d. find volume, mass, centre of mass and moment of inertia using cylindrical coordinates.

e. demonstrate certain conclusions relating to cylindrical coordinates.

Reading

Read Section 16.9, pages 1092-1097.

Exercises

Complete problems 1-10, 15-21, 23, 25, 27, 29, 30 and 32-35 on pages 1097-1098.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

cylindrical coordinates

equations relating rectangular coordinates to cylindrical coordinates

equations relating cylindrical coordinates to rectangular coordinates

cylindrical coordinate surfaces

cylindrical wedge

formula relating triple integral of rectangular coordinates to triple integral of cylindrical coordinates

volume formula expressed in cylindrical coordinates

Skills

Before you proceed to Objective 10, make certain that you can meet each of the sub-objectives listed under Objective 9.

Objective 10

After completing this section, you should be able to

a. convert an equation expressed in rectangular coordinates into one expressed in spherical coordinates.

b. convert an equation expressed in spherical coordinates into one expressed in rectangular coordinates.

c. evaluate triple integrals of spherical coordinates.

d. find volume, mass, centre of mass and moment of inertia using spherical coordinates.

e. demonstrate certain conclusions relating to spherical coordinates.

Reading

Read Section 16.10, pages 1098-1105.

Exercises

Complete odd-numbered problems 1-19, and problems 20-28, 34, 35 and 37 on pages 1105-1106.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

formula relating the triple integral of a basic solid to the limit of a Riemann sum: $\iiint_T f(x, y, z)dx\ dy\ dz = \lim_{\text{diam } T_i \to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*, z_i^*)(\text{volume of } T_i)$

spherical coordinates

longitude

colatitude

polar angle

equations relating rectangular coordinates to spherical coordinates

equations relating spherical coordinates to rectangular coordinates

spherical wedge

formula relating triple integral of rectangular coordinates to triple integral of cylindrical coordinates

volume formula expressed in spherical coordinates

Skills

Before you proceed to Objective 11, make certain that you can meet each of the sub-objectives listed under Objective 10.

Objective 11 After completing this section, you should be able to

a. express coordinates in the *xy*-plane as coordinates in the *uv* plane.

b. find the Jacobian matrix of a transformation.

c. express the area of a region in the *xy*-plane as a double integral of the corresponding region in the *uv*-plane.

d. express the double integral of a continuous function in the *xy*-plane as a double integral of the corresponding region in the *uv*-plane, and solve.

e. express the triple integral of a continuous function in the *xyz*-space as a triple integral of the corresponding region in the *uvw*-space, and solve.

f. use Jacobians to solve problems of volume, mass, centre of mass and moment of inertia.

g. demonstrate various conclusions relating to Jacobians.

Reading

Read Section 16.11, pages 1106-1111.

Exercises

Complete odd-numbered problems 1-17, and problems 19-33 on pages 1111-1112.

Terms

Make certain that you can define, and use in context, the terms, concepts and formulas listed below.

Jacobian matrix

Jacobian transformation of coordinates: two-dimensional case

Jacobian transformation of coordinates: three-dimensional case

Skills

Before you proceed to Unit 6, make certain that you can meet each of the sub-objectives listed under Objective 11.